Building a Library of Mechanized Mathematical Proofs

Why do it?

And what is it like to do?

Rob Arthan

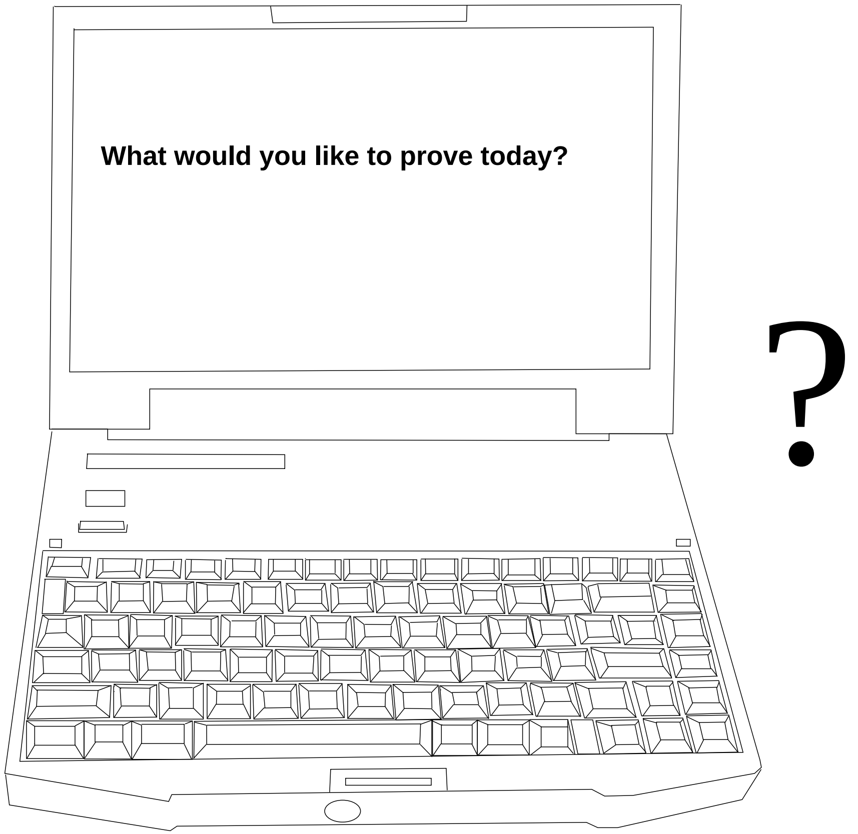
Lemma 1 Ltd. / Queen Mary, University of London

ICMS 2010

14th September, 2010

A Hypothetical Question

What would Newton or the Bernoullis or Gauss or Delaunay or Hilbert or . . . your list goes here . . . have done with one of these:



Formalised Mathematics ca. 1910: $1+1=$ hmmm?

"The properties of $\dot{2}$ are largely analogous to those of 1, while the properties of 2_r are more analogous to those of 2."

A.N. Whitehead and B. Russell *Principia Mathematica *56* **p. 375** 1910

What about properties of e or π or $\zeta(1)$ or $\pi_{13}(S^5)$ or $\ldots$?

- The immense symbolic processing tasks defy human capabilities.
- But 100 years on we have machines to do all that.
- Machines don't know what symbols to process.
- Synergy between human and computer is what's called for.

Pure Mathematician: Why bother?

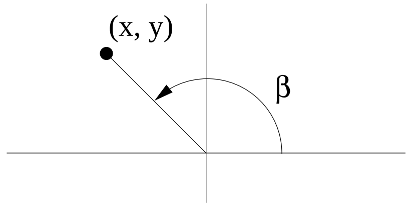
- Mathematical arguments do contain errors!
- How many in the classification of finite simple groups?
- Even Aigner & Ziegler's *"Proofs from THE BOOK"* (ed. 1).

  See Bill Casselman. *The Difficulties of Kissing in Three Dimensions.*

- Avoid controversies about use of computers:
  - 4 colour theorem
  - Kepler sphere packing conjecture
- Eliminate circle-squarers, cube-doublers, angle-trisectors!

. . . or even serious, but misguided "provers" of $\mathsf{P} = \mathsf{NP}$

Engineer: Why bother?



A simple example: calculating bearings

- Specification: take $\beta_{(x,y)}$ chosen in the appropriate quadrant where:

$$\mathsf{tan}\ \beta_{(x,y)} = \frac{y}{x}$$

- Design: calculate the bearing using: $\beta = \mathsf{tan}^{-1}(y/x)$
- Code: `beta = atan(y/x)`
- BUT: $\beta_{(1,1)} = \pi/4 \neq 5\pi/4 = \beta_{(-1,-1)}$

Testing tends not to find subtle errors.

Computer scientist: Why bother?

Computer scientists...

- ... prove theorems
- ... specify, design and code programs

See above!

Proposal

We should use computers to support mathematical endeavours.

- I will describe one approach that offers:
  - high assurance ($1 \neq 2$)
  - extensiblity and programmability
  - access to a large library of existing results
- In the rest of the talk I hope to:
  - Show how the LCF paradigm offers high assurance
  - Give a flavour of what it is like to develop a theory like the calculus in an LCF-style system

A Deductive System

Judgments: $m \ \mathsf{D} \ n$, $m, n \in \mathbb{Z}$ (intended meaning: m divides n)

Rules:

$$\frac{}{m \ \mathsf{D} \ m} : \text{axiom}, m \neq 0$$

$$\frac{m \ \mathsf{D} \ n}{m \ \mathsf{D} \ -n} : - \qquad \frac{m \ \mathsf{D} \ n_1 \quad m \ \mathsf{D} \ n_2}{m \ \mathsf{D} \ n_1 + n_2} : +$$

A deduction:

$$\dfrac{\dfrac{}{1 \ \mathsf{D} \ 1} : \text{axiom} \qquad \dfrac{\dfrac{}{1 \ \mathsf{D} \ 1} : \text{axiom}}{1 \ \mathsf{D} \ -1} : -}{1 \ \mathsf{D} \ 0} : +$$

- Robin Milner's LCF method implements a deductive system as a data type.
- Strongly typed programming language enforces the rules.

The Deductive System in ML

Demo 1

```
local
  datatype THEOREM = D of (int * int);
in
  type THEOREM = THEOREM;
  infix D; infix ++;
  exception NOT_ALLOWED;

  fun axiom m = if m <> 0 then m D m else raise NOT_ALLOWED;
  fun -- (m D n) = m D ~n;
  fun (m1 D n1) ++ (m2 D n2) =
    if m1 = m2 then m1 D (n1 + n2) else raise NOT_ALLOWED;
end;
```

A Decision Procedure

Demo 1 concluded.

Given `m` and `n`, the function `decide` tries to prove that `m` divides `n`:

```
fun decide m n =
  if n < 0 then --(decide m (~n))
  else if n <= m then axiom m
  else decide m (n-m) ++ axiom m;
```

Logical kernel will not allow invalid deductions:

```
> decide 2 6;
val it = 2 D 6 : THEOREM
> decide 2 7;
val it = 2 D 8 : THEOREM
> decide 0 0;
Exception- NOT_ALLOWED raised
```

A Real System: ProofPower-HOL

Demo 2.

- Member of the HOL family implemented for industrial use.

  Cf. Classic HOL (Gordon), HOL IV (Slind, Norrish), HOL Light (Harrison).

- Expressions and predicates represented by the type $TERM$, entered using “Quine corners”: $\ulcorner 1 + 2 \urcorner$, $\ulcorner 1 = 2 \urcorner$

- Abstract data type of theorems is the type THM. Printed with a turnstile: $\vdash \neg \mathit{1} = \mathit{2}$.

- Extensive facilities for programming with syntax.

- Maintains a database of theories containing specifications (i.e., defining properties of types and constants) and theorems.

- Powerful higher-level tools for automated and interactive proof.

Characteristics of the LCF Approach

- Logical kernel is small and simple to check
  — possibly even formally verifiable.
- Derived rules may fail to produce desired output but can't produce unsound results.
- Can safely code very complex algorithms.
- Potential for cross-checking using alternative compilers and/or implementations.
- There are other complementary approaches.

Definitions In ProofPower-HOL

- Syntax for defining new constants (including functions, functionals etc.) with an arbitrary defining property:

$$
\begin{array}{|l}
\boldsymbol{<NAME>} : <TYPE> \\
\hline
<DEFINING\ PROPERTY>
\end{array}
$$

- Proof obligation to verify consistency (often discharged automatically).

HOL Constant

$$
\begin{array}{|l}
\boldsymbol{Even} : \mathbb{N}\ SET \\
\hline
\forall n \bullet\ n \in Even \Leftrightarrow (\exists m \bullet n = 2*m)
\end{array}
$$

HOL Constant

$$
\begin{array}{|l}
\boldsymbol{Odd} : \mathbb{N}\ SET \\
\hline
\forall n \bullet\ \neg\ 2*n \in Odd \land 2*n+1 \in Odd
\end{array}
$$

Calculus In ProofPower-HOL 1

- Write $(f\ Deriv\ c)\ x$ to mean function f has derivative c at x (i.e., $df/dx = c$ or $f'(x) = c$ in the usual vernacular).

$$\$\boldsymbol{Deriv} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \to \mathbb{R} \to BOOL$$

$$\begin{aligned} &\forall f\ c\ x\bullet\ (f\ Deriv\ c)\ x \\ \Leftrightarrow\ \ &\forall e\bullet\ 0. < e \Rightarrow \exists d\bullet\ 0. < d\ \wedge \\ &\quad \forall y\bullet\ Abs(y{-}x) < d \wedge \neg y{=}x \Rightarrow Abs((f\ y{-}f\ x)/(y{-}x)\ -\ c) < e \end{aligned}$$

- This definition is trivially consistent.
- All the usual theorems: product rule, chain rule, Rolle, IVT, MVT, ...

Calculus In ProofPower-HOL 2

- Define the exponential function by the differential equation:

$$\boldsymbol{Exp} : \mathbb{R} \to \mathbb{R}$$

$$Exp\ 0. = 1. \land (\forall x \bullet (Exp\ Deriv\ Exp\ x)\ x)$$

- Prove the consistency via theory of power series and differentiation of limits.
- Logarithm defined as left inverse of exponential.
- All the usual basic theorems, e.g.,

$$\vdash \forall\ x \bullet\ 0. < x \Rightarrow (Log\ Deriv\ x^{-1})\ x : THM$$

- Similar treatment of trigonometric functions.

Calculus In ProofPower-HOL 3

Demo 2 continued.

- Integration via the Kurzweil-Henstock gauge integral. FTC, areas, ...
- A theorem from antiquity:

$$\vdash \forall\ r\bullet\ 0.<r \Rightarrow$$
$$\{(x,\ y) | Sqrt\ (x \ \hat{}\ 2 + y \ \hat{}\ 2) \le r\}\ Area\ \pi * r \ \hat{}\ 2$$

- A theorem of Minkowski:

$$\vdash \forall A\ a\bullet$$
$$A \in Convex \wedge A \in Bounded \wedge \neg A = \{\}$$
$$\wedge\ (\forall x\ y\bullet\ (x,\ y) \in A \Rightarrow (\sim x,\ \sim y) \in A)$$
$$\wedge\ A\ Area\ a \wedge a > 4.$$
$$\Rightarrow \exists i\ j : \mathbb{Z}\bullet\ (\mathbb{ZR}\ i,\ \mathbb{ZR}\ j) \in A \wedge \neg(\mathbb{ZR}\ i,\ \mathbb{ZR}\ j) = (0.,\ 0.)$$

- de Bruijn factor. Formal / Informal. Typically 0.5 – 5?
- See Freek Wiedijk's web site `http://www.cs.ru.nl/~freek/`

Example: A Combinatorics Problem

Demo 2 concluded.

- $(m+n)^m$ should give the number of ways of drawing m samples with replacement out of a set of $m+n$ elements.
- *DistinctSamples* n m should give the number of ways of drawing m samples without replacement out of a set of $m+n$ elements.

$$\boldsymbol{DistinctSamples} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$

$$\begin{aligned} & (\forall n \bullet\ \mathit{DistinctSamples}\ n\ 0 = 1) \\ \wedge\ & (\forall m\ n \bullet \\ & \mathit{DistinctSamples}\ n\ (m+1) = (n+m+1) * \mathit{DistinctSamples}\ n\ m) \end{aligned}$$

- Plan: give high assurance solution to a combinatorics problem by:
  - 1) Proving that these functions do give the desired results.
  - 2) Symbolically executing them inside the theorem prover.

Final Remarks

- Machine-checked proof has a part to play.
- Technology can help: critical bugs are not inevitable!
- Formalisation need not lead to combinatorial explosion.
- Lots of fascinating work to do.

Thank you!

For the slides, go to: `http://www.lemma-one.com/papers/`

Links

- For ProofPower
  `http://www.lemma-one.com/ProofPower/papers/index.html`
- Other implementations of HOL:
  `http://hol.sourceforge.net/`
  `http://www.cl.cam.ac.uk/research/hvg/Isabelle/`
  `http://www.cl.cam.ac.uk/~jrh13/hol-light/`
- The Flyspeck project:
  `http://code.google.com/p/flyspeck/`
- Freek Wiedijk's progress report on 100 theorems:
  `http://www.cs.ru.nl/~freek/100/index.html`
- The Mizar project:
  `http://mizar.uwb.edu.pl/`

The following is the Standard ML source of the demos:

Text dumped to file 82dedsys.ML

```
local
  datatype THEOREM = D of (int * int);
in
  type THEOREM = THEOREM;
  infix D; infix ++;
  exception NOT_ALLOWED;

  fun axiom m = if m <> 0 then m D m else raise NOT_ALLOWED;

  fun -- (m D n) = m D ~n;

  fun (m1 D n1) ++ (m2 D n2) =
    if m1 = m2 then m1 D (n1 + n2) else raise NOT_ALLOWED;
end;
```

Text dumped to file 82s1.ML

```
1+2;
fun fact n = if n <= 0 then 1 else n * fact(n-1);
fact 1s0;
map fact [1, 2, 3, 4, 5, 6, 7];
use "82dedsys.ML";
axiom 1;
1 D 1;
3 D 5;
val thm1 = axiom 1;
val thm2 = -- (axiom 1);
val thm3 = thm1 ++ thm2;
fun decide m n =
  if n < 0 then --(decide m (~n))
  else if n <= m then axiom m
  else decide m (n-m) ++ axiom m;
```

```
decide 1 1;
decide 1 ~1;
decide 2 6;
decide 13 1001;
decide 2 7;
decide 0 0;
decide ~1 1;
```

Text dumped to file 82s2.ML

```
val tm1 = ⌜0 * 1⌝;
val ty1 = type_of tm1;
val (f1, args1) = strip_app tm1;
val ty_f1 = type_of f1;
val thm1 = get_spec f1;
val thm2 = list_∀_elim [ ⌜1⌝, ⌜1⌝ ] thm1;
val thm3 = ∧_left_elim thm2;
val thm4 = rewrite_conv[] ⌜(0+1+2+3)*(3+2+1+0)⌝;
set_goal([], ⌜∀m i j : ℕ• m^(i + j) = m^i * m^j⌝);
a( REPEAT strip_tac );
a(induction_tac ⌜j⌝);
a(rewrite_tac[ ℕ_exp_def ]);
a(asm_rewrite_tac[ plus_assoc_thm1 , ℕ_exp_def ]);
a( PC_T1 "lin_arith" prove_tac [] );
val thm5 = pop_thm();
val tm2 = ⌜(λx•x + 1)⌝;
val ty2 = type_of tm2;
val thm6 = rewrite_conv[]⌜(λx•x + 1) 2⌝;
val tm3 = ⌜[x; y; 1]⌝;
val ty3 = type_of tm3;
val (f3, args3) = strip_app ⌜[x; y; z]⌝;
val ty_f3 = type_of f3;
val thm7 = rewrite_conv[nth_def] ⌜Nth [x; y; z] 2⌝;
val thm8 = get_spec⌜Map⌝;
val ty4 = type_of⌜Map⌝;
```

```
val thm9 = rewrite_conv[map_def] ⌜Map (λx• x + 1) [1; 2; 3]⌝;
val tm4 = ⌜2 ∈ {1; 2; 3}⌝;
val (f4, args4) = strip_app tm4;
val ty_f4 = type_of f4;
val thm4 = get_spec f4;
val thm10 = prove_rule[] tm4;
val thm11 = pc_rule1 "sets_ext1"prove_rule[elems_def]
    ⌜Elems [1; 2; 3] = {1; 2; 3}⌝;
get_spec⌜π⌝;
get_spec⌜Sin⌝;
get_spec⌜$Area⌝;
get_spec⌜$Int_R⌝;
get_spec⌜Gauge⌝;
get_spec⌜TaggedPartition⌝;
get_spec⌜$Fine⌝;
get_spec⌜RiemannSum⌝;
distinct_samples_def;
val thm12 = (print "\n\n"; conv_rule(MAP_C plus_conv)(pure_rewrite_conv
    [distinct_samples_def,
    pc_rule1 "lin_arith" prove_rule[]⌜∀m•m*1 = m⌝,
    plus_assoc_thm1]
    ⌜DistinctSamples 10 (0+1+1+1)⌝));
val thm13 = rewrite_rule[] thm12;
distinct_samples_finite_size_thm;
samples_finite_size_thm;
(* and then .... *)
use_file"82demo2proof.ML";
val thm14 = pop_thm();
get_spec⌜Elems⌝;
get_spec⌜Distinct⌝;
get_spec⌜Finite⌝;
```

Text dumped to file 82demo2proof.ML

```
set_goal([], ⌜ (* Try not to show from HERE ... *)
```

```
    let S = {L | Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23}
    in let  X = {L | L ∈ S ∧ ¬L ∈ Distinct}
    in  S ∈ Finite
    ∧  ¬#S = 0
    ∧  X ⊆ S
    ∧  #X / #S > 1/2
⌝);
a(rewrite_tac[let_def]);
a(strip_asm_tac(∀_elim⌜365⌝ range_finite_size_thm1));
a(lemma_tac⌜23 ≤ #{i | 1 ≤ i ∧ i ≤ 365}⌝ THEN1 asm_rewrite_tac[]);
a(all_fc_tac[distinct_samples_finite_size_thm]);
a(all_fc_tac[samples_finite_size_thm]);
a(REPEAT_N 2 (POP_ASM_T(ante_tac o ∀_elim⌜23⌝)) );
a(POP_ASM_T discard_tac THEN strip_tac THEN strip_tac);
a(pure_asm_rewrite_tac[conv_rule(ONCE_MAP_C eq_sym_conv)ℕℝ_one_one_thm,
    ℕℝ_ℕ_exp_thm,
    ℝ_frac_def]);
a(asm_tac(rewrite_conv[]⌜ℕℝ 365 ^ 23⌝));
a(PC_T1"predicates" rewrite_tac[] THEN strip_tac
    THEN1 (pure_asm_rewrite_tac[ℕℝ_one_one_thm]
        THEN PC_T1 "lin_arith" prove_tac[]));
a(REPEAT strip_tac THEN1 PC_T1 "sets_ext1" prove_tac[]);
a(pure_rewrite_tac[ℕℝ_one_one_thm]);
a(LEMMA_T⌜{L | (Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23)
        ∧ ¬L ∈ Distinct}
 = {L | Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23} \
    {L | Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23
        ∧ L ∈ Distinct}⌝
    pure_rewrite_thm_tac
    THEN1 PC_T1 "sets_ext1" prove_tac[]);
a(lemma_tac⌜{L | Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23 ∧
        L ∈ Distinct} ⊆
    {L | Elems L ⊆ {i | 1 ≤ i ∧ i ≤ 365} ∧ # L = 23}⌝
    THEN1 PC_T1 "sets_ext1" prove_tac[]);
a(ALL_FC_T (MAP_EVERY (ante_tac o
    once_rewrite_rule[conv_rule(ONCE_MAP_C eq_sym_conv)ℕℝ_one_one_thm]))
```

```
    [size_⊆_diff_thm]);
a(pure_rewrite_tac[ℕℝ_plus_homomorphism_thm,
    pc_rule1 "ℝ_lin_arith" prove_rule[]
    ⌜∀a b c:ℝ•a = b + c ⇔ b = a − c⌝]
    THEN STRIP_T pure_rewrite_thm_tac);
a(LIST_DROP_NTH_ASM_T [3, 6, 9] pure_rewrite_tac);
a(LEMMA_T ⌜∀a b c d:ℝ•ℕℝ 0 < b ∧ ℕℝ 0 < d ∧ a∗d < b∗c ⇒ a/b < c/d⌝
    bc_thm_tac
    THEN1 (REPEAT strip_tac
        THEN1 ALL_FC_T1 fc_⇔_canon asm_rewrite_tac[ℝ_cross_mult_less_thm]));
a(pure_asm_rewrite_tac[ℕℝ_ℕ_exp_thm,
    pc_rule1 "ℝ_lin_arith" prove_rule[]
    ⌜∀a b c:ℝ•a < ℕℝ 2 ∗ (b − c) ⇔ a + ℕℝ 2 ∗ c < ℕℝ 2 ∗ b⌝,
    REPEAT_C (once_rewrite_conv[distinct_samples_rw_thm]
        THEN_C rewrite_conv[])
    ⌜DistinctSamples (365−23) 23⌝]);
a(pure_rewrite_tac[ℕℝ_plus_homomorphism_thm1,
    ℕℝ_times_homomorphism_thm1,
    ℕℝ_less_thm]);
a(rewrite_tac[]) (∗ ... down to HERE! ∗);
```